Kinetic Equations

Text of the Exercises

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Exercise 1

We recall that the collision term of the (general) Boltzmann equation for hard sphere interactions is:

$$Q\left(f,f\right)\left(v\right) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\left(f\left(v'\right)f\left(v'_*\right) - f\left(v\right)f\left(v_*\right)\right)\left|\left(v-v_*\right)\cdot\omega\right| d\omega\ dv_*. \tag{1}$$

We consider now an homogeneous solution f of the Boltzmann equation (which does not depend on the position variable x) and radial in velocity (which depends only on the norm $|v|$ of the velocity variable v).

- Under those hypotheses, show that the collision term (1) of the Boltzmann equation writes:

$$\begin{aligned} &Q\left(f,f\right)\left(v\right) = \\ &= 4\pi^2\int_0^{+\infty}\int_0^{\pi}\int_0^{\pi}\left(f\left(\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\right)f\left(\sqrt{r_1^2\sin^2\theta_1 + r^2\cos^2\theta}\right) - \right.\\ &\qquad \left. - f\left(r\right)f\left(r_1\right)\right)\left|r_1\cos\theta_1 - r\cos\theta\right|\sin\theta\sin\theta_1 r_1^2 d\theta\ d\theta_1\ dr_1, \end{aligned} \tag{2}$$

  where r denotes $|v|$.

  Hint: Denote as r_1 the norm of the velocity v_*, θ the angle between the velocity v and ω, and θ_1 the angle between the velocity v_* and ω.

- Considering the transformation $\varphi : (r, r_1, \theta, \theta_1) \mapsto (r', r'_1, \theta', \theta'_1)$ defined through the system:

$$\begin{cases} r'\cos\theta' = r_1\cos\theta_1, \\ r'\sin\theta' = r\sin\theta, \\ r'_1\cos\theta'_1 = r\cos\theta, \\ r'_1\sin\theta'_1 = r_1\sin\theta_1, \end{cases} \tag{3}$$

  show that the collision term (2) can be abbreviated as:

$$C\int_0^{+\infty}\int_0^{\pi}\int_0^{\pi}\Big(f(t,r')f(t,r'_1) - f(t,r)f(t,r_1)\Big)V(r,r_1,\theta,\theta_1)r_1^2 d\theta d\theta_1 dr_1, \tag{4}$$

  with $V(r,r_1,\theta,\theta_1) = \big|r_1\cos\theta_1 - r\cos\theta\big|\sin\theta\sin\theta_1$.

Exercise 2

We consider now the gain term of the collision operator, that is the part:

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} f'f'_* B(v-v_*,\omega)d\omega dv_*$$

in the right-hand side of the Boltzmann equation. In the case of hard sphere interactions with a solution which is homogeneous and radial in velocity, we saw that this term can be written as:

$$J(f) = \int_0^{+\infty}\int_0^{\pi}\int_0^{\pi} f\Big(t, \sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\Big) f\Big(t, \sqrt{r_1^2\sin^2\theta_1 + r^2\cos^2\theta}\Big) \times \Big|r_1\cos\theta_1 - r\cos\theta\Big| \sin\theta\sin\theta_1 r_1^2 d\theta d\theta_1 dr_1. \quad (5)$$

- Considering $x = \cos\theta$ and $y = \cos\theta_1$, show that (5) is equal to

$$2\int_0^{+\infty}\int_0^{1}\int_0^{1} f\Big(t, \sqrt{r^2 - r^2x^2 + r_1^2y^2}\Big) f\Big(t, \sqrt{r_1^2 - r_1^2y^2 + r^2x^2}\Big) \times \Big(|r_1y - rx| + |r_1y + rx|\Big) r_1^2 dy dx dr_1. \quad (6)$$

- Considering $u = \sqrt{r^2 - r^2x^2 + r_1^2y^2}$ and $v = \sqrt{r_1^2 - r_1^2y^2 + r^2x^2}$, show that (5) is equal to

$$4\int_0^{+\infty}\int_0^{+\infty} f(t,u)f(t,v)G(r,u,v)uv du dv, \quad (7)$$

where G is defined as:

$$\begin{cases} G(r,u,v) &= 0 & \text{if } u^2+v^2 \leqslant r^2, \\ G(r,u,v) &= 1 & \text{if } u \geqslant r,\ v \geqslant r, \\ G(r,u,v) &= v/r & \text{if } u \geqslant r,\ v \leqslant r, \\ G(r,u,v) &= u/r & \text{if } u \leqslant r,\ v \geqslant r, \\ G(r,u,v) &= \sqrt{u^2+v^2-r^2}/r & \text{if } u^2+v^2 \geqslant r^2,\ u \leqslant r,\ v \leqslant r. \end{cases}$$

Exercise 3

Finally, we consider the loss term of the collision operator, that is the part:

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} ff' B(v - v_*, \omega) d\omega dv_*$$

in the right-hand side of the Boltzmann equation. In the case of hard sphere interactions with a solution which is homogeneous and radial in velocity, we saw that this term can be written as:

$$\int_0^{+\infty}\int_0^{\pi}\int_0^{\pi} f(t,r)f(t,r_1)V(r,r_1,\theta,\theta_1)r_1^2 d\theta d\theta_1 dr_1 = f(t,r)L(f)(t,r),$$

with

$$L(f)(t,r) = \int_0^{+\infty} P(r,r_1)f(t,r_1)r_1^2 dr_1,$$

and

$$P(r,r_1) = \int_0^{\pi}\int_0^{\pi} \Big|r_1\cos\theta_1 - r\cos\theta\Big| \sin\theta\sin\theta_1 d\theta d\theta_1. \quad (8)$$

Show that the quantity P in (8) can be expressed as:

$$P(r,r_1) = \big(2r + \frac{2r_1^2}{3r}\big)\mathbb{1}_{r_1 \leqslant r} + \big(2r_1 + \frac{2r^2}{3r_1}\big)\mathbb{1}_{r_1 > r}.$$